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# Numerical Treatment of Wave Propagation in Layered Media

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Computational tools for the analysis of wave motion in layered media are reviewed. While frequency-domain-based procedures have been most widely used, recent developments by the authors and their co-workers have shown promise with regard to efficiency and accuracy in calculations directly in the time domain. These maintain the existing framework for layered-media modeling and exploit the hyperbolic nature of the governing wave equations. High-order approximations of the dispersion relation, amenable to straightforward numerical implementation, and ingredients of an adaptive scheme empowered with robust error estimation are outlined toward improved treatment of wave motion.

## INTRODUCTION

The study of wave propagation in layered media is an integral part of dynamic soil-structure interaction investigations as well as procedures for geotechnical site characterization. In all cases of practical interest, numerical modeling of layered media is a requirement in estimation of the dynamic stiffness of foundations and calculation of soil-structure system response to incident seismic waves. The same is true in extraction of soil properties from surface-wave data and other measurements.

Computations of dynamic response of soil-structure systems are usually carried out on the basis of models similar to the generic arrangement depicted in Fig. 1. Two complementary regions can be distinguished: the *interior,* i.e., a neighborhood of the structure encompassing heterogeneities, irregularities and nonlinearities, and the *exterior,* typically a horizontally layered medium extending to great, usually assumed infinite, distance from the structure. Finite elements have been the most common choice of interior discretization by virtue of their versatility in dealing with the complexity of this region. On the other hand, the exterior

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has been represented by means of a "transmitter" or "absorber" placed on the boundary of the interior. This transmitting or absorbing boundary simulates the propagation of waves in the exterior and has been derived using a variety of schemes and approximations, with finite elements as well as boundary elements. Absorbers are classified (Kausel 1988, Givoli 1991) as *local* or *global* (nonlocal) depending on the extent of the constraint they impose on wave motion in space and time. Local absorbing boundary conditions couple only nearby spatial locations and time stations. They are computationally efficient (as they lead to sparse systems of equations) but can produce spurious reflections. Global conditions constrain the entire boundary and all past time points. They are computationally demanding but accurate, typically limited only by discretization error.

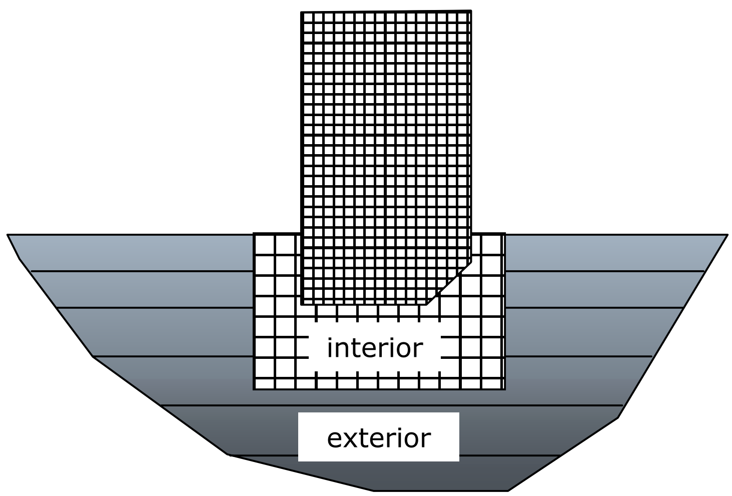



**Figure 1.** Partitioning of a typical soil-structure system: interior and exterior

## HORIZONTALLY LAYERED MEDIA

Modeling of the exterior as a horizontally layered medium has been routine in soil-structure interaction studies. A variety of alternative numerical formulations have been considered in this direction. The present paper is focused on one such approach that has been in wide use and shows promise for future developments and enhancements. Referring to a layered medium in a two-dimensional setting as depicted in Fig. 2, semidiscretization, i.e., discretization only with respect to the *vertical* coordinate *z*, is applied:

$$\mathbf{u}(x,z,t) \approx \sum_i \mathbf{N}_i(z)\mathbf{U}_i(x,t)$$

Note that **u** is the displacement vector, and $\mathbf{N}_i$ and $\mathbf{U}_i$ are nodal interpolation functions of the vertical coordinate $z$ and displacement vectors, respectively, associated with layer-to-layer interfaces $z = z_i$ (see Fig. 2). A similar expression can be written for general three-dimensional cases. Using the Principle of Virtual Work with respect to the *vertical* direction, a system of wave equations is obtained:

$$\mathbf{A}\mathbf{U}_{xx} + \mathbf{B}\mathbf{U}_{x} - \mathbf{G}\mathbf{U} - \mathbf{M}\mathbf{U}_{tt} = \mathbf{0} \qquad (1)$$

It is worth mentioning that **A**, **B**, **G** and **M** are banded matrices (see Tassoulas 1981, Guddati 1998, Park 2000). In applications to date, layer (material) properties have been assumed independent of horizontal location (range independence). The matrices **A**, **B**, **G** and **M** are then constant. Clearly, the system (Eq. 1) involves one spatial (horizontal) variable (two in three dimensions) and, thus, recognizes explicitly the intrinsic contrast between horizontal and vertical directions in the medium.

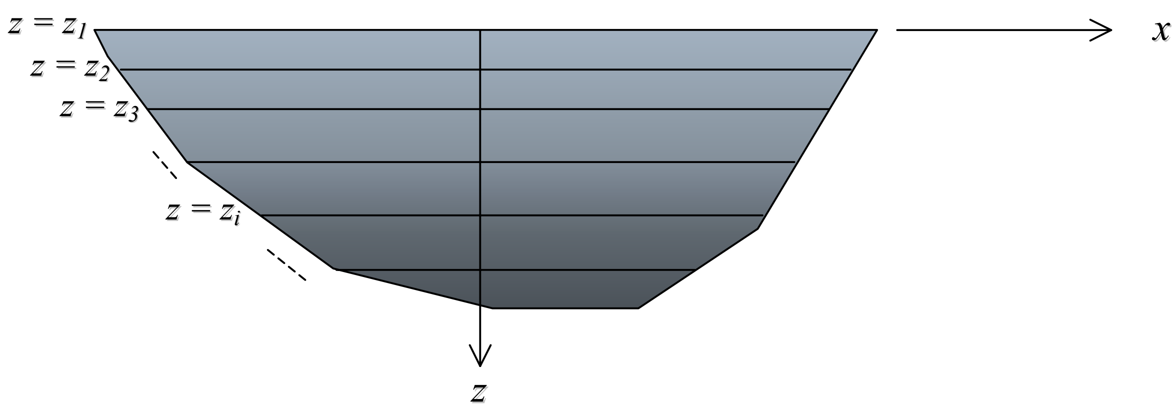



**Figure 2.** Vertical discretization of a layered medium.

## CONSISTENT TRANSMITTING BOUNDARY

One of the most efficient uses of the approach outlined above has been in the development of a "consistent transmitting boundary" for the representation of the exterior in dynamic soil-structure interaction computations (Waas 1972, Kausel 1974). It is based on a frequency-domain solution of the governing equations (Eq. 1). One looks for harmonic modes of the form:

$$\mathbf{U} = \mathbf{V}e^{-ikx}e^{i\omega t} \qquad (2)$$

For any given frequency $\omega$, the wave-number-wave-mode pairs ($k$, $\mathbf{V}$) satisfy the quadratic eigenvalue problem:

$$(k^2\mathbf{A} + ik\mathbf{B} + \mathbf{G} - \omega^2\mathbf{M})\mathbf{V} = \mathbf{0} \tag{3}$$

The consistent transmitting boundary is obtained by expressing the wave motion in the exterior as a combination of admissible modes and calculating the so-called dynamic stiffness matrix relating boundary forces and displacements. It is worth mentioning that this boundary constitutes a perfect (global) absorber (as long as the underlying assumptions of linearity and range-independence prevail). As such, it has been used extensively in calculations of dynamic stiffness of foundations and soil-structure system response. Extensions and applications of the basic ideas have been pursued successfully in many other directions, including earthquake response of dam-water-sediment-rock systems (Lotfi et al. 1987, Bougacha and Tassoulas 1991a, b), dynamics of poroelastic media (Bougacha et al. 1993a, b), and interpretation of nondestructive tests of pavements (Foinquinos et al. 1995), to mention a few.

## TIME-DOMAIN COMPUTATIONS

As alluded to above, the consistent transmitting boundary is a frequency-domain-based development and, therefore, not convenient or practical in problems involving nonlinearities. An equivalent boundary for direct use in the time domain has been elusive. This is the focus of recent research by the authors and their co-workers. Several time-domain alternatives have been explored successfully while maintaining the ideas underlying the general approach outlined for layered media, i.e., semidiscretization combined with the Principle of Virtual Work with respect to the vertical direction. Time-domain tools for exterior modeling have been formulated by exploiting the hyperbolic character of the governing equations and proceeding toward direct solution in space (horizontal direction) and time. Alongside these procedures, arbitrarily-high-order approximations of the dispersion relation have been obtained, amenable to implementation within the existing computational framework, while adaptive time-domain schemes involving error estimation and enrichment of the exterior representation have been devised. These developments are summarized below.

## CHARACTERISTICS METHOD

Toward a more general global boundary condition for time-domain computations, a characteristics-based solution procedure (Guddati and Tassoulas 1998a, 1998b) has been developed, applicable to antiplane shear-wave propagation in homogeneous media (layer and half/full space). The procedure involves three steps: (a) utilizing semidiscretization as described above to reduce the governing partial differential equation (PDE), into a system of hyperbolic PDEs in a single spatial variable and time; (b) splitting the displacement vector into wave modes, each satisfying the scalar dispersive wave equation; (c) solving the resulting scalar equation on the characteristic grid shown in Fig. 3. The third step is the key to the efficiency of the characteristics method. The solution in the space-time domain facilitates an element-by-element solution; the elements are processed in the order specified in Fig. 3 using a cell-centered finite-difference scheme. Such an element-by-element treatment makes the computational cost comparable to that of explicit time-marching, yet the method is unconditionally stable, owing to the space-time orientation of the grid.

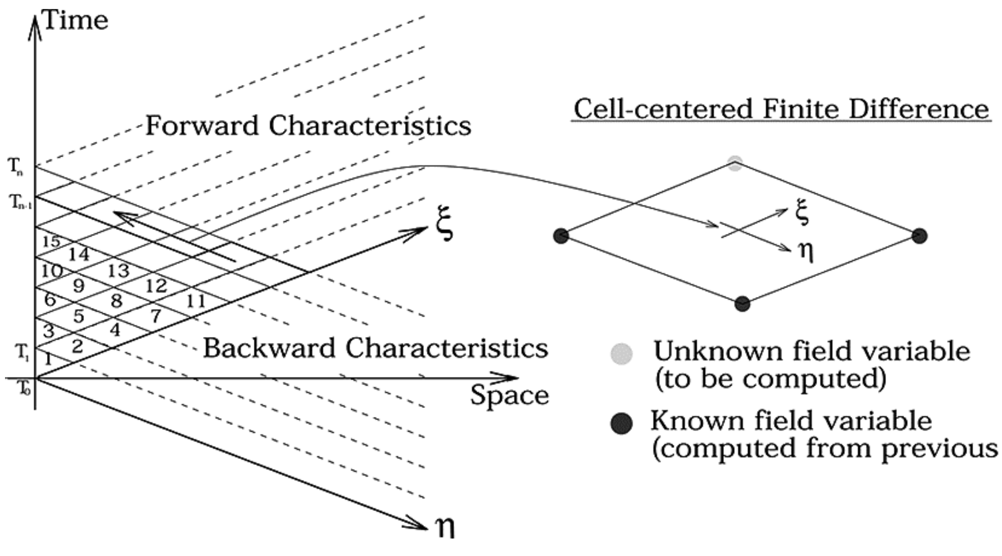



**Figure 3**. Space-time discretization for the characteristics method: element-by-element computation is the key to efficiency.

The characteristics-based computation for the exterior results in the following form of the absorbing boundary condition relating the vector of boundary nodal forces **S** (discrete equivalent of tractions) and the vector of boundary nodal displacements **U**:

$$\mathbf{S} = \mathbf{K}\mathbf{U} + \mathbf{C}\frac{\partial \mathbf{U}}{\partial t} + \mathbf{H}(t) \tag{4}$$

It is worth mentioning that **K** and **C** are symmetric positive definite matrices, and $\mathbf{H}(t)$ is a vector evaluated at every time step from the characteristic grid. The positive definite nature of **K** and **C** facilitates unconditionally stable coupling with interior time-stepping schemes such as the Newmark $\beta$ method. With respect to accuracy, the characteristics boundary condition is comparable to the convolution-based methods. The computational cost of characteristic boundary, on the other hand, is at least an order of magnitude less than that of the convolution-based methods. In addition, it has been shown that the characteristics boundary significantly outperforms local absorbers with respect to overall accuracy and stability (Guddati, Park and Tassoulas, 1999), especially for fluid-structure interaction problems.

## SPACE-TIME DISCONTINUOUS GALERKIN METHOD

The space-time discontinuous Galerkin method (STDGM) (Park and Tassoulas 2002) provides global absorbing conditions for layered media, or, in general, for systems characterized by multiple wave speeds. An overall solution procedure involving the STDGM can be sketched as follows. After semidiscretization of a medium extending to infinity in the *x*–direction, the exterior domain becomes a region in the $x-t$ space bounded by the characteristic associated with the highest wave speed *c* of the system (Fig. 4). The domain can then be partitioned by horizontal lines (parallel to the *x*-axis) and inclined lines (parallel to the fastest characteristic) into parallelograms and triangles, which constitute the space-time elements to which the discontinuous Galerkin method (Johnson 1987) is applied in an element-by-element fashion. That is, in order to obtain the solution at time $t_n$, parallelogram elements $(n,1)$, $(n,2), \ldots, (n,n-1)$ are processed first, and the absorbing condition for the interior is derived as the interaction force $\mathbf{F}_c$ acting on the interior while the influence of the interior on the exterior is evaluated as the force $\mathbf{F}_\infty$ acting on triangle $(n,n)$. As the evaluation of $\mathbf{F}_c$ relies on the field variables defined on triangle $(n,n)$ and $\mathbf{F}_\infty$ on the solutions from the interior at $t_n$, the system equations for the interior and triangle $(n,n)$ are coupled and thus solved simultaneously.

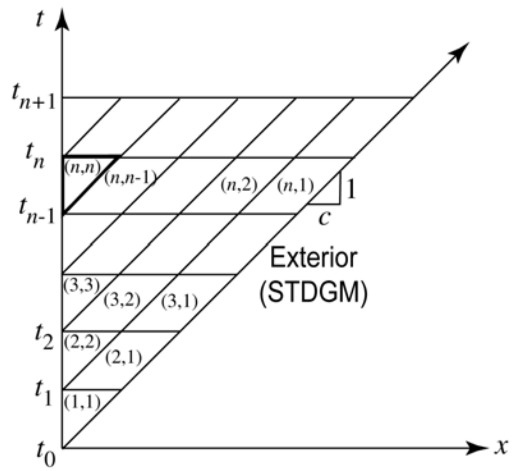



**Figure 4**. Exterior representation by the space-time discontinuous Galerkin method

When the excitation is prescribed as seismic ground motion, the problem on the exterior domain is formulated in terms of the relative motion (with respect to the free-field ground motion), and matching conditions are imposed on the boundary between the interior and exterior domains. This modification introduces effective seismic forces in defining $\mathbf{F}_c$ and $\mathbf{F}_\infty$ allowing for the consideration of soil-structure systems subjected to seismic waves (Park and Antin 2004).

With regard to computational effort, the STDGM is comparable to the characteristics method, more efficient by at least an order of magnitude than methods based on convolution. Furthermore, all computational experiments to date suggest that the current implementation (Park and Tassoulas 2002) is second-order accurate and unconditionally stable. These features make the STDGM a reliable choice for modeling unbounded layered media in soil-structure interaction analysis in the time domain.

## APPROXIMATION OF DISPERSION RELATION: CONTINUED FRACTION ABSORBERS AND ARBITRARILY WIDE-ANGLE WAVE EQUATIONS

In spite of the excellent savings presented by the space-time methods, some large-scale problems still cannot be solved efficiently using global methods. Furthermore, in many large-scale problems, the motion is predominantly composed of propagating (traveling) waves. Local absorbing boundary conditions (ABCs) can be used to simulate such waves in unbounded domains. Although hierarchies of increasingly accurate local ABCs have been available (e.g., Engquist and Majda 1979), they could not be implemented in conventional computational settings due to the presence of high-order derivatives (see, e.g., Givoli 1992). Recently, arbitrarily accurate local ABCs that can be implemented in standard finite-element and finite-difference settings have been developed, first for layers with straight computational boundaries (Guddati 1998; Guddati and Tassoulas 2000), and later for general polygonal computational domains (Lim 2003; Guddati and Lim 2004). Both the derivation and implementation of these ABCs are based on the continued-fraction approximation of the dispersion relation, and thus, they are named continued-fraction ABCs. Discretization of the boundaries, combined with the interior problem, results in a system of integrodifferential equations in time:

$$\mathbf{M}\frac{\partial^2\mathbf{U}}{\partial t^2}+\mathbf{C}\frac{\partial\mathbf{U}}{\partial t}+\mathbf{KU}+\mathbf{R}\int\mathbf{U}dt=\mathbf{F}, \quad (5)$$

The coefficient matrices **C** and **R** are contributed by the ABCs. All the coefficient matrices, **M**, **C**, **K** and **R**, in the above equation are sparse, symmetric and positive definite. The Newmark $\beta$ time-integration schemes are generalized to solve the above integrodifferential system in an effective manner (Guddati and Lim 2004).

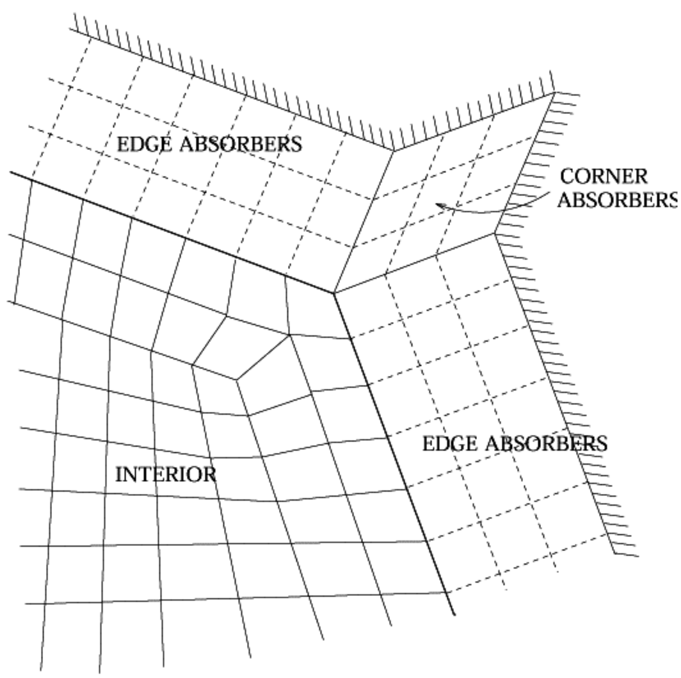
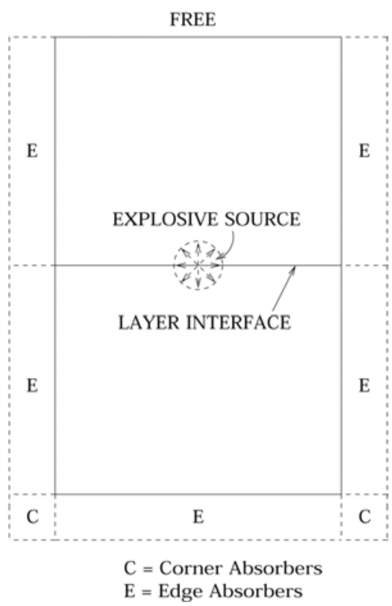
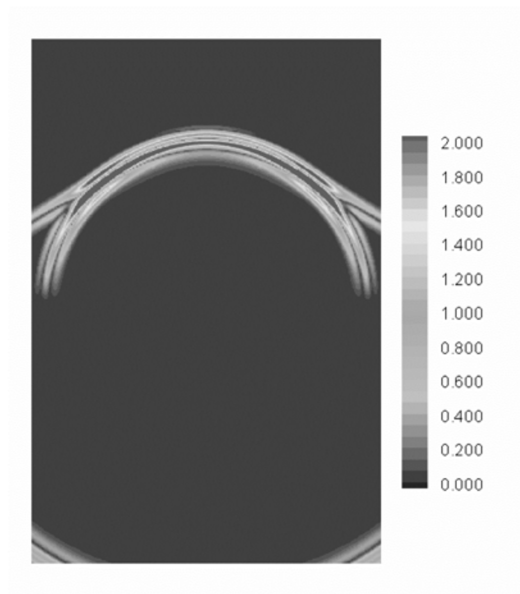



**Figure 5**. Implementation of continued-fraction absorbers: for most practical problems, only three to four layers of absorbing elements are needed.

The striking feature of the continued-fraction boundary condition is the ease of its implementation, which involves just adding a few (three to four) layers of absorbing elements for each edge and a consistent parallelogram mesh for each corner (see Fig. 5). It follows that the added computational cost is negligible in comparison with that required for the interior.

To illustrate the effectiveness of the proposed technique, representative results are shown in Fig. 6, where antiplane shear waves from an explosion in a layered half-space are analyzed. Three layers of edge absorbers are added at the bottom and on both sides, and consistent corner absorbers at the bottom two corners. Fig. 6-b clearly indicates that the boundary condition is effective in absorbing all body and head waves.




**(a)** **(b)**

**Figure 6.** Effectiveness of the continued-fraction absorbing boundary conditions.

Extension of continued-fraction absorbers for plane-strain conditions is not straightforward as the dispersion relation for the elastic wave equation involves matrix operators. Guddati (2004) has recently developed an effective way of devising a matrix continued-fraction approximation, which has been successfully used in deriving absorbing boundary conditions for P, S and Rayleigh waves. The implementation is almost identical to that for antiplane shear waves. Extension of continued-fraction absorbers to layered elastic media and to dispersive waves is currently underway.

The continued-fraction approximation of the dispersion relation has also led to new one-way wave equations named Arbitrarily Wide Angle Wave Equations (AWWE) that have applications in subsurface imaging. In the context of seismic imaging, for example, waves are sent from the surface and the reflected waves from oil reservoirs (referred to as reflectors) are measured. The resulting surface trace is processed (migrated) to estimate the location of the reflectors. AWWE have been utilized for such processes. A representative result is shown in Fig. 7, where a synthetic surface trace, obtained from exploding reflector computation (Claerbout 1985), is processed with AWWE migration to form the image of the interior. The results clearly indicate the effectiveness of AWWE imaging (for details, see Guddati and Heidari 2004).

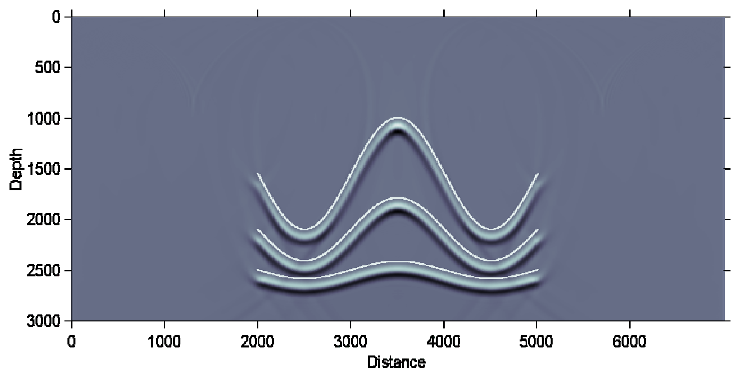



**Figure 7**. Subsurface imaging using AWWE: the white lines are the exact locations of the reflectors while the underlying grayscale images are the AWWE results. Note that the exact reflectors have been shifted upwards for the purpose of facilitating comparison with the computed ones.

## ADAPTIVE TIME-DOMAIN COMPUTATIONS

Use of local absorbing conditions provides an attractive means of simulating wave motion in unbounded media by virtue of their superior computational efficiency compared with global conditions. However, local conditions prescribed on the boundary of truncation $\Gamma_\infty$ are bound to introduce errors in the computed solution due to their inability to perfectly absorb outgoing waves, which cause spurious wave reflection from $\Gamma_\infty$ (Fig. 8). Although highly accurate local conditions of arbitrarily high order are being developed (e.g., Givoli and Neta 2003, Guddati and Tassoulas 2000), the magnitude of the ensuing errors cannot easily be predicted and, normally, a number of numerical experiments as well as experience are called for to verify the adequacy of the solution. Consequently, the entire analysis procedure can be time-consuming and the results may still be unreliable.

The idea of adaptive computations (Park 2004) is based on the observation that, given the size of a computational domain, there is a local absorbing condition of lowest order within the hierarchy of conditions under consideration that can produce the solution of desired accuracy. On the other hand, although higher-order conditions undoubtedly lead to greater accuracy, lower-order conditions can also produce equally accurate results in the region of interest at the expense of enlarging the computational domain (thus placing the absorbing boundary farther from the region of interest). Consequently, given the order of the absorbing condition, it should be possible to select the smallest size of the computational domain that can produce the solution of desired accuracy. Use of such a condition or domain can result in optimal treatment of the problem in the sense that a desired result is obtained with the least amount of effort.

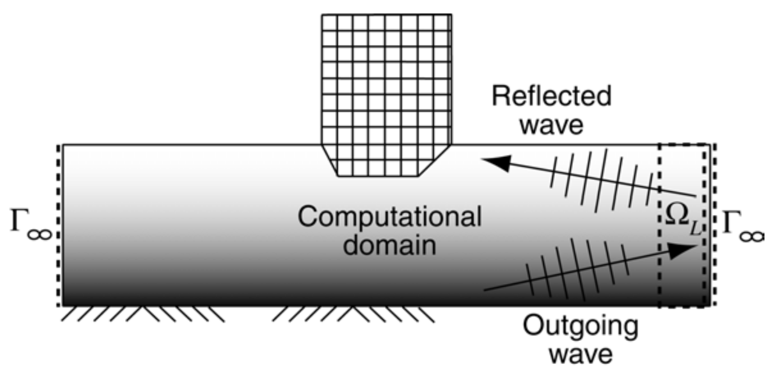



**Figure 8**. Reflected waves caused by the use of local absorbing conditions

The foregoing arguments allow one to design an adaptive analysis procedure, which would select the optimal order of the absorbing condition or optimal size of the domain based on some a posteriori error estimation techniques to control the magnitude of the spurious wave reflected from the boundary of truncation, thus, ultimately, controlling the accuracy of the solution in the region of interest. Such a procedure would be significantly more efficient and reliable because, first, local boundaries are used and thus the computational cost is very low; and second, the optimal order of or the optimal size of the domain is automatically determined by the solution algorithm without manually driven numerical experiments.

Adaptive computations in the time domain would require a hierarchy of absorbing conditions of arbitrarily high order, if the order of the condition is to be controlled, and a method to extract the reflected wave field from the computed solution. Extraction of the reflected wave should be achieved by a solution procedure applied to a small domain $\Omega_L$ in the vicinity of the boundary $\Gamma_\infty$ (Fig. 8). Based on this local solution procedure, an error estimator can be defined that has good correlation with the intensity of reflected waves. It is the responsibility of the overall adaptive scheme to use this estimator to control the order of the absorbing condition. An implementation of the procedure in the context of one-dimensional wave propagation is presented by Park (2004). Certainly, the challenge is to extend the concept to multidimensional situations.

## CONCLUSIONS

Recent progress in the direction of time-domain computations of wave motion in layered media has led to a suite of effective tools. Space-time methods, including the *characteristics method* and the *space-time discontinuous Galerkin method,* exploit the hyperbolic character of the governing equations and lead to efficient, accurate and stable schemes. Furthermore, *continued-fraction absorbing boundary conditions* and *arbitrarily-wide-angle wave equations* have been devised. These approximations of the dispersion relation are highly efficient and accurate, particularly attractive in large-scale problems, and can be implemented easily within the existing computational framework. Also, *adaptive* time-domain procedures involving error estimation and enrichment of the exterior representation are being devised. The numerical treatment of waves in layered media is indispensable to studies of dynamic soil-structure interaction and geotechnical site characterization, and the time-domain developments reviewed in this paper are promising and versatile alternatives to frequency-domain-based procedures, especially in problems involving nonlinearities.

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